

On local-hidden-variable no-go theorems

André Allan Méthot

Département d'informatique et de recherche opérationnelle
Université de Montréal, C.P. 6128, Succ. Centre-Ville
Montréal (QC), H3C 3J7 CANADA
`methotan@iro.umontreal.ca`

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Abstract

The strongest attack against quantum mechanics came in 1935 in the form of a paper by Einstein, Podolsky and Rosen. It was argued that the theory of quantum mechanics could not be called a complete theory of Nature, for every element of reality is not represented in the formalism as such. The authors then put forth a proposition: we must search for a theory where, upon knowing everything about the system, including possible hidden variables, one could make precise predictions concerning elements of reality. This project was ultimately doomed in 1964 with the work of Bell, who showed that the most general local hidden variable theory could not reproduce correlations that arise in quantum mechanics. There exist mainly three forms of no-go theorems for local hidden variable theories. Although almost every physicist knows the consequences of these no-go theorems, not every physicist is aware of the distinctions between the three or even their exact definitions. Thus we will discuss here the three principal forms of no-go theorems for local hidden variable theories of Nature. We will define Bell theorems, Bell theorems without inequalities and pseudo-telepathy. A discussion of the similarities and differences will follow.

1 Introduction

In 1935, Einstein, Podolsky and Rosen (EPR) wrote their famous paper “Can quantum-mechanical description of physical reality be considered complete?” [1], wherein the authors answered the question in the negative. Completeness was then considered an important criterion regarding the validity of any physical theory of Nature, and would still be for any proponent of determinism and a classical view of the universe. Thus, their paper was a great threat to the validity of quantum mechanics (QM).

Completeness is defined in the EPR paper as “every element of the physical reality must have a counterpart in the physical theory”, where physical reality should be interpreted as “If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.” EPR used the correlations obtained from bipartite measurements of an entangled state to claim that position and momentum can, do, and in fact must have simultaneous realities. More precisely, they used a bipartite entangled state and separated the two subsystems into space-like-separated regions. In their particular case, if one was to perform a position measurement on the first subsystem, from the result we could determine the exact position of the second subsystem. The same could be done for momentum. From their definition

of physical reality and since, according to relativity, the space-like separation of the two subsystems prevents any interaction between them, EPR concluded that position and momentum must have simultaneous realities. From the fact there are no interactions between the subsystems, they also concluded that these realities existed all along, from the time of the separation of the subsystems. Since the formalism of QM precludes such a description of reality, they were forced to conclude that QM cannot be considered a complete theory of Nature. For EPR, Nature possesses local hidden variables (LHVs), that can or cannot be known, which determine the behavior of a system under any measurement. In this picture, a measurement only reveals a pre-existing property of Nature, while the usual consensus in quantum mechanics is that a measurement forces a property of the system into existence. Einstein then spent the rest of his life in search of such a theory.

The LHV paradigm is the straightforward mathematical representation of local realism. In a LHV model, the hidden variables are set according to some probability distribution at the creation of a state. They can only be accessed experimentally through measurement, which perturbs the state and possibly alters the hidden variables. If we were to know the actual values of the hidden variables, we could fully predict the behaviour of the system under any measurement of an element of reality, but this lack of knowledge forces to average over the possible values of the LHVs. Thus the Heisenberg uncertainty principle is not violated and the probabilistic structure of QM is preserved. The locality criterion means that the outcome of a measurement on space-like separated systems cannot be correlated in any way other than through the original hidden variables, which are fixed once the state is created and change only according to local operations.

Even though the EPR paper contained logical errors [2], it was still the firmest attack against a quantum description of the physical world. Even the mightiest and hardest defender of QM, Bohr [3], was not able to firmly put the EPR argument to rest [2]. It took the better part of three decades for a complete refutation of the EPR argument to be put forth: in [4], Bell laid down the most general LHV model for a particular measurement setup. He then showed that the expectation value of the measurement operator could not, in *any* LHV model, come near the actual value predicted by QM. Experiments have confirmed the correctness of the predictions of QM [5, 6].

The work of Bell has been called “the most profound discovery of science” [7]. At least, it could easily be argued that Bell’s theorem has changed our view of Nature in the same way as Newton’s classical mechanics and Einstein’s relativity has. From his work, we now know that conjugate operators do not have simultaneous existence and only a measurement can force a state which is not an eigenstate of the operator to bring a value to the physical quantity attributed to the operator into existence.

In this paper, we present different forms of no-go theorems concerning LHV theories. To the knowledge of the author, only three forms of refutation exist, and a comprehensive comparative study has not yet been published. It should be noted that this paper does not have the goal of being an exhaustive survey of all the LHV no-go theorems. In Section 2, we will give formal definitions to the three forms of no-go theorems, namely Bell theorems, Bell theorems without inequalities¹ and pseudo-telepathy. Section 3 will then contain a discussion of the similarities and differences of the three forms.

2 The three forms

2.1 Bell theorems

The first proof that the physical world could not be described by any LHV theory came from Bell in 1964 in the form of an inequality [4]. Bell bounded the absolute value of the expectation value of a specific operator

¹Also called Bell inequalities without probabilities, Bell inequalities without inequalities or all-versus-nothing refutation of EPR.

in *any* LHV model and showed that quantum mechanics violates this bound. As such, we define a Bell theorem as follows.

Definition 1 (Bell theorem). *A Bell theorem is a set of multipartite measurements on an entangled state where the correlations obtained from the measurements cannot be reproduced by any local classical model where no communication between the participants is allowed.*

2.1.1 An example of a Bell theorem

The example we are to discuss here is often thought as the generic Bell theorem and was put forth as an experimental proposal by Clauser, Horne, Shimony and Holt [8]. Suppose A_1 and A_2 are measurement settings on Alice’s apparatus and B_1 and B_2 are measurement settings on Bob’s. Given that the value of the outcome to each measurement lies between -1 and 1 , we can easily bound the value of the operator $\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle$ in any LHV theory. Since the equation is linear in every variable and since every variable must be independent of one another (locality constraint) the maximum will be reached with every variable taking an extremal value, 1 or -1 . Therefore, we have $|\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle| \leq 2$. In quantum mechanics, we can find appropriate measurements on a singlet state, $(|+-\rangle - |-+\rangle)/\sqrt{2}$, to yield $|\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle| = 2\sqrt{2}$. The details are left as an exercise to the reader. From the fact that no LHV model can obtain a higher expectation value than 2 , while QM can, is a clear proof that quantum correlations cannot be simulated by a LHV theory. If we now take the predictions of QM to be correct, we have to forsake the search for a local realistic description of reality.

2.2 Bell theorems without inequalities

A Bell theorem appeals to a statistical argument and, as such, does not look attractive to many. Thus, a more direct rebuttal of LHVs was sought for. The first such proof came from Heywood and Redhead [9], where the authors aimed to propose an experimental verification of the Kochen-Specker theorem [10] to reject locality². It is important to note that Bell inequalities without inequalities are not experimental proposals which would rule out any LHV model in only one run. As Asher Peres once said [11]: “The list of authors [who has made this mistake] is too long to give explicitly, and it would be unfair to give only a partial list.” We will discuss this in more detail in Section 3.1.

Definition 2 (Bell theorems without inequalities). *A Bell theorem without inequalities is a set of multipartite measurements on an entangled state where any local classical model, which is to attempt to simulate the probability distribution of the outputs given by quantum mechanics, will attribute a non zero probability to a measurement outcome that is forbidden by quantum mechanics or will never produce certain outcomes which are predicted with a non zero probability in quantum mechanics.*

2.2.1 An example of a Bell theorem without inequalities

For this example, we will give Brassard’s [13] rendition of Hardy’s proof [14]. Let us start with the state $(|--\rangle + |+-\rangle + |-+\rangle)/\sqrt{3}$ along the z axis. Let say that Alice and Bob are now given the choice of performing either the σ_z or σ_x measurement. According to QM, if Alice and Bob are to measure $\sigma_x \otimes \sigma_x$, then they will receive the output $--$ with probability $1/12$. Let us now assume that the state has LHVs that will produce a $--$ output on a $\sigma_x \otimes \sigma_x$ measurement. From the criteria of locality and realism, we now have that any local σ_x measurement on this particular state will produce the output $-$. Let us now see what

²The first example is often wrongly attributed to Greenberger, Horne and Zeilinger [12].

happens if Alice and Bob are to measure $\sigma_x \otimes \sigma_z$ or $\sigma_z \otimes \sigma_x$. From the predictions of QM, the state should never be allowed to produce $--$ as output. Once again according to the criteria of locality and realism, we are forced to conclude that upon a local σ_z measurement, the state will produce $+$ as output. Therefore, a $\sigma_z \otimes \sigma_z$ measurement on this particular instance is bound to output $++$, which is forbidden by QM. So in order for the LHV theory to output $--$ on a $\sigma_x \otimes \sigma_x$ measurement with a non zero probability, it will also output $++$ on a $\sigma_z \otimes \sigma_z$ measurement with non zero probability.

2.3 Pseudo-telepathy

Pseudo-telepathy was first defined, although not yet termed as such, in a paper by Brassard, Cleve and Tapp, where they turned the Deutsch-Jozsa algorithm into a distributed form to show that an exponential number of bits of communication was needed to simulate the correlations of a certain number of singlet states [15]. For a complete survey, please refer to [16].

Definition 3 (Game). A bipartite game $G = (X, Y, R)$ is a set of inputs $X = X^{(A)} \times X^{(B)}$, a set of outputs $Y = Y^{(A)} \times Y^{(B)}$ and a relation $R \subseteq X^{(A)} \times X^{(B)} \times Y^{(A)} \times Y^{(B)}$.

Definition 4 (Winning Strategy). A winning strategy for a bipartite game $G = (X, Y, R)$ is a strategy according to which for every $x^{(A)} \in X^{(A)}$ and $x^{(B)} \in X^{(B)}$, Alice and Bob output $y^{(A)}$ and $y^{(B)}$ respectively such that $(x^{(A)}, x^{(B)}, y^{(A)}, y^{(B)}) \in R$.

Definition 5 (Pseudo-telepathy). We say that a bipartite game G exhibits pseudo-telepathy if bipartite measurements of an entangled quantum state can yield a winning strategy, whereas no classical strategy that does not involve communication is a winning strategy.

The extension to the multipartite case is trivial. The generalization of Definition 5 can be translated into a set of multipartite measurements on an entangled state where any local classical model which is to attempt to produce outputs that are not forbidden by quantum mechanics will fail. We refer to this phenomenon by the term pseudo-telepathy because of its shortness and its illustrativeness. To someone who has no knowledge of quantum mechanics, and thus still believes in local realism, these correlations between the measurement outcomes can only be explained by some communication between the sub-systems. Thus if we make these measurements in space-like separated regions the sub-systems act as if they could telepathically communicate instantaneously.

2.3.1 An example of a pseudo-telepathy game

We present here the pseudo-telepathy game generally known as the Magic Square game [17]. The participants, namely Alice and Bob, are each presented with a question: a random trit $x^{(A)} \in \{0, 1, 2\}$ and $x^{(B)} \in \{0, 1, 2\}$ respectively. They must produce three bits each, $y_1^{(A)}, y_2^{(A)}, y_3^{(A)}$ and $y_1^{(B)}, y_2^{(B)}, y_3^{(B)}$ respectively. In order for them to win, $y_1^{(A)} + y_2^{(A)} + y_3^{(A)}$ must be even, $y_1^{(B)} + y_2^{(B)} + y_3^{(B)}$ must be odd and $y_{x^{(B)}}^{(A)}$ must equal $y_{x^{(A)}}^{(B)}$. In order for them to have a winning strategy without resorting to QM or communicating, Alice and Bob must share a 3×3 table of 0s and 1s such that the sum of the elements in each row is even and the sum of the elements in each column is odd. A simple parity argument shows that such a table is impossible. On the other hand, if Alice and Bob are allowed to share entanglement, they can find a strategy which, while convoluted, can be explicitly constructed. It should be noted that they cannot construct the 3×3 table required classically, but they can use the correlations of the quantum state to give three bits each respecting the above conditions.

3 Discussion of the similarities and differences

3.1 Similarities

The obvious similarity between the three forms of no-go theorems is that they all reject a local realistic description of Nature. Therefore, the physical world cannot be described by any LHV theory. As mentioned in Section 1, all three require many runs of the same experiment, with settings chosen at random on each run, to rule out any LHV model. A LHV might be lucky and answer correctly for many runs. Since a LHV theory can only succeed with a marginal probability of success, but still can succeed, we can only collect overwhelming evidence against LHVs. The only rejection we can make in one run is of quantum mechanics itself. If we consider perfect apparatus, then the production of a forbidden output by quantum mechanics, in a Bell theorem without inequalities or pseudo-telepathy experiment, would invalidate the correctness of quantum mechanical predictions.

It is also interesting to remark that every pseudo-telepathy game is a Bell theorem without inequalities and every Bell inequality without inequalities is a Bell theorem. These facts follow from Definition 1, 2 and 5. In pseudo-telepathy, the fact that no LHV strategy can always output within the relation R can be seen as giving a non zero probability to an output that is forbidden in quantum mechanics—quantum mechanics being able to never give this output. And from Definition 2 and Definition 5, we clearly have a set of measurements on an entangled state where the correlations obtained from the measurements cannot be reproduced by any LHV model, hence falling into Definition 1.

3.2 Differences

Bell theorems and pseudo-telepathy are in a sense only quantitatively different. Pseudo-telepathy is simply an inequality where the quantum violation of the inequality reaches the maximal algebraic value. We saw in Section 2.3.1 an example where we appeared not to be concerned with reaching the maximal value of an inequality, but even this example can be converted as an inequality. The exercise is left to the reader.

It might be tempting to think that Bell theorems without inequalities are the same as pseudo-telepathy. In fact, the list of people who have made this error might also be too long to include here. The reason is that in both of these paradigms, we are concerned with proving a no-go theorem by a contradiction. However, it was proven recently that no pseudo-telepathy game can exist where the participants share only one entangled qubit, a 2×2 system [18]. For pseudo-telepathic correlations to arise we need at least a 3×3 system [9] or a $2 \times 2 \times 2$ system [20]. Therefore, Hardy's state cannot yield correlations strong enough for pseudo-telepathy. As a consequence, pseudo-telepathy and Bell theorems without inequalities are different paradigms. The difference is buried subtly in Definition 2 and Definition 5. In the first case, we require the LHV model to be able to generate all the possible outputs according to QM without outputting a forbidden output, while in the second case the requirement is weaker; we only need to produce outputs that can be generated by QM. In other words, in pseudo-telepathy, we only need to avoid forbidden outputs, but we do not need to produce every possible output. Hardy's proof relies on the fact that once in a while the experiment will generate $++$ on a $\sigma_x \otimes \sigma_x$ measurement, but for a pseudo-telepathy game it is of no help.

4 Conclusion

We have seen the formal definition of the three forms of LHV no-go theorems and how they differ one from another. We have thus shown that it is important to look closely at our models when we want to describe nature, for we could have seen that there is a qualitative difference between Hardy's no-go theorems and most

of the other Bell theorems without inequalities. In this hierarchy of no-go theorems, pseudo-telepathy is the stronger refutation of the local realistic viewpoint as it lies at the top of the LHV no-go theorem hierarchy. Not every Bell theorem is a Bell theorem without inequalities and not every Bell theorem without inequalities is a pseudo-telepathy game, while the converse is true. There exists states that can generate correlations strong enough for a Bell theorem without inequalities while the same state cannot yield a pseudo-telepathy game [18].

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